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2008 J. Phys. A: Math. Theor. 41 164021

(http://iopscience.iop.org/1751-8121/41/16/164021)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **41** (2008) 164021 (8pp) doi:10.1088/1751-8113/41/16/164021

On the Casimir effect with general dispersion relations

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Received 22 October 2007, in final form 23 October 2007
Published 9 April 2008
Online at stacks.iop.org/JPhysA/41/164021

Abstract
We investigate the Casimir force in one dimension for general dispersion relations. In particular, we find an explicit formula for the operator that maps general dispersion relations into the corresponding Casimir force functions. We show that this operator is a modified Laplace transform and we verify that it is independent of the regularization method used in its calculation. The knowledge of this operator allows one, in particular, to identify the circumstances under which the behaviour of the dispersion relation at one length scale can affect the behaviour of the Casimir force at a different length scale.

PACS numbers: 12.20.−m, 12.20.Ds, 11.55.Fv

1. The Casimir force and dispersion relations

There are numerous influences on the strength and sign of the Casimir effect, see [1–4], such as the surfaces' shape, roughness and reflection properties as well as the temperature. Here, we will explore aspects of how the nontrivial propagation of the quantum field in a medium influences the Casimir effect. Concretely, we will study the influence of general non-vacuum dispersion relations on the Casimir effect. We will here not focus on any particular modified dispersion relation. Instead, our aim is to calculate the operator which maps generic dispersion relation functions $\omega(k)$ into the resulting Casimir force functions $F(L)$. This then allows us to investigate, in particular, how a nontrivial behaviour of a dispersion relation at one length scale affects the behaviour of the Casimir force function at another length scale.

To this end, for simplicity, we will consider a massless real scalar field in one dimension between two perfectly conducting parallel 'plates', which in this case are of course mere points. We place these points at $x = 0$ and $x = L$, i.e., we impose Dirichlet boundary conditions $\phi(0, t) = 0 = \phi(L, t)$ for all t. The usual bosonic quantization procedure then yields the one-particle Hilbert space $\mathcal{H} = L^2([0, L])$ on which is built the symmetric Fock space $\mathcal{F}_+(\mathcal{H})$ with Fock vacuum $|\Omega\rangle$ so that the Hamiltonian H expressed in terms of creation and annihilation operators reads

$$H = \sum_{n=0}^{\infty} \omega(k_n) \left(a_n^* a_n + \frac{1}{2} \right), \qquad k_n = \frac{n\pi}{L}. \tag{1}$$

In the vacuum the dispersion relation is linear, $\omega(k) = k$, and the vacuum energy $\langle\Omega|H|\Omega\rangle$ between plates of distance L is therefore divergent, $E(L) = \frac{1}{2}\sum_{n=0}^{\infty}\omega(k_n) = \infty$, with the modes appearing to contribute the more the shorter their wavelength, i.e., the larger k and n are. One proceeds by regularizing the divergence and by then calculating the change in the regularized total energy (of a large region that contains the plates) when varying L. As is well known, the resulting expression for the Casimir force remains finite after the regularization is removed and reads

$$F(L) = -\frac{\pi}{24L^2}. \tag{2}$$

It has been shown that this result does not depend on the choice of regularization method. Our aim now is to recalculate the Casimir force within standard quantum field theory while allowing general nonlinear dispersion relations. To this end, it will be convenient to write generalized dispersion relations in the form

$$\omega(k) = k_c f\left(\frac{k}{k_c}\right), \tag{3}$$

where $k_c > 0$ is a constant with the units of momentum and where the function f encodes the nonlinearities. We shall make the following minimal assumptions:

- $f(0) = 0$, and $f(x) = x + o(x)$ as $x \searrow 0$ (regular dispersion at low energies);
- $f(x) \geqslant 0$ when $x \geqslant 0$ (stability: each mode carries positive energy).

We will use the term dispersion relation for both $\omega(k)$ and $f(x)$.

For generically modified dispersion relations the vacuum energy must be assumed to be divergent and therefore in need of regularization. Let us therefore regularize the energy by introducing an exponential cut-off function, parametrized by $\alpha > 0$. (We will later show that other cut-off functions could be used if more convenient or if necessary to ensure convergence for a given dispersion relation.) The regularized vacuum energy between the plates then reads

$$E_\alpha(L) = \frac{1}{2}\sum_{n=0}^{\infty} k_c f(\kappa_n)\, \mathrm{e}^{-\alpha k_c f(\kappa_n)}, \tag{4}$$

with $\kappa_n = n\pi/k_c L$. The outside region being infinitely extended, the energy density outside the plates can be calculated from the above by letting L go to infinity:

$$\mathcal{E}_\alpha = \lim_{L\to\infty} \frac{E_\alpha(L)}{L} = \frac{k_c^2}{2\pi}\int_0^\infty \mathrm{d}x\, f(x)\, \mathrm{e}^{-\alpha k_c f(x)}. \tag{5}$$

In the second step we used (4) and that the limit is a Riemann sum with $\Delta x = L^{-1}$.

In order to calculate the Casimir force, let us now consider a very large but finite region, say of length M, which contains the two plates. The total regularized energy in this region is finite and consists of the energy between the plates (4), plus the energy density outside the plates (5), multiplied by the size of the region outside, namely $M - L$. Note that choosing M large enough ensures that the energy density outside the plates does not depend on L. The regularized Casimir force is the negative derivative of the total energy with respect to a change in the distance of the plates:

$$F_\alpha(L) = -\frac{\partial}{\partial L} E_\alpha(L) + \mathcal{E}_\alpha. \tag{6}$$

Hence, before removing the regularization (i.e. before letting $\alpha \searrow 0$), the Casimir force in the presence of a nonlinear dispersion relation is given by

$$F_\alpha(L) = \frac{k_c}{2L}\sum_{n=0}^{\infty}\varphi_\alpha(n) + \mathcal{E}_\alpha. \tag{7}$$

Here, we introduced the following notation:

$$\varphi_\alpha(t) = \kappa_t f'(\kappa_t)\, \mathrm{e}^{-\alpha k_c f(\kappa_t)}\,(1 - \alpha k_c f(\kappa_t))\,. \tag{8}$$

We observe that if the first term in (7) were an integral instead of a series, then the force function would identically vanish. This suggests the use of the Euler–Maclaurin sum formula,

$$\sum_{a<n\leqslant b}\xi(n) = \int_a^b \xi(t)\mathrm{d}t + \sum_{r=0}^{k}\frac{(-1)^{r+1}B_{r+1}}{(r+1)!}(\xi^{(r)}(b) - \xi^{(r)}(a)) + \frac{(-1)^k}{(k+1)!}\int_a^b B_{k+1}(t)\xi^{(k+1)}(t)\,\mathrm{d}t, \tag{9}$$

to express the series of φ_α as an integral of φ_α plus correction terms, the latter summing up to the Casimir force. We perform the obvious substitutions, note that $\lim_{x\to\infty}\varphi_\alpha^{(n)}(x) = 0$ for all $n \geqslant 0$ and that $\varphi_\alpha(0) = 0$ to obtain

$$F_\alpha(L) = -\frac{k_c}{2L}\sum_{r=1}^{k}\frac{B_{2r}}{2r!}\varphi_\alpha^{(2r-1)}(0) + \frac{k_c}{2L}\Omega_k[\varphi_\alpha], \tag{10}$$

where $\Omega_k[\varphi_\alpha]$ denotes the remainder integral

$$\Omega_k[\varphi_\alpha] = \frac{(-1)^k}{(k+1)!}\int_0^\infty B_{k+1}(t)\varphi_\alpha^{(k+1)}(t)\,\mathrm{d}t, \tag{11}$$

and where we used that, except for B_1, all Bernoulli numbers B_s with odd indices s are zero. As noted earlier, the integral approximating the sum and that of equation (5) cancelled out. The actual Casimir force is obtained by removing the regularization:

$$F(L) = \lim_{\alpha\searrow 0} F_\alpha(L). \tag{12}$$

Explicitly, let us calculate this for a polynomial dispersion relation $f(x) = \sum_{s=0}^{n}\nu_s x^s$. Since $\varphi_\alpha(t)$ is jointly infinitely differentiable in α and t, the limit $\alpha \searrow 0$ in $\varphi_\alpha(t)$ can be taken before differentiating. From (8) we then have $\varphi_0(t) = \lim_{\alpha\searrow 0}\varphi_\alpha(t) = \kappa_t f'(\kappa_t)$. Thus, iterated differentiation yields

$$\varphi_0^{(n)}(t)|_{t=0} = n\left(\frac{\pi}{k_c L}\right)^n f^{(n)}(0). \tag{13}$$

After sufficiently many differentiations, the remainder integral $\Omega_k[\varphi_\alpha]$ vanishes. Indeed, firstly for all fixed $t > 0$, $\varphi_\alpha^{(k+1)}(t) \to 0$ as $\alpha \searrow 0$, and secondly $\varphi_\alpha^{(k)}(t) \to 0$ as $t \to \infty$. Split the integral $\int_0^\infty = \int_0^b + \int_b^\infty$ for some finite $b > 0$. The first integral can be exchanged with the limit and vanishes by the above remark. The second can be estimated by

$$\left|\int_b^\infty B_{k+1}(t)\varphi_\alpha^{(k+1)}(t)\,\mathrm{d}t\right| \leqslant |B_{k+1}|\left|\varphi_\alpha^{(k)}(b)\right| \xrightarrow{\alpha\searrow 0} 0, \tag{14}$$

where we used the second remark and the fact that the Bernoulli functions are bounded above by the corresponding Bernoulli numbers. Finally, $f^{(s)}(0) = s!\nu_s$. The Casimir force is thus given by

$$F(L) = -\frac{k_c}{2L}\sum_{r=1}^{k}\frac{(2r-1)B_{2r}}{2r}\nu_{2r-1}\left(\frac{\pi}{k_c L}\right)^{2r-1}. \tag{15}$$

For example, for the vacuum dispersion relation $f(x) = x$ we recover the well-known result $F(L) = -\pi/24L^2$ as it should be. Interestingly, the even powers in a polynomial dispersion relation, i.e. the coefficients ν_{2r}, do not contribute to the Casimir force. We also note that the mapping $\mathcal{K}: \; f(x) \mapsto F(L)$ from polynomial dispersion relations into their corresponding Casimir force functions is a linear operator between the two sets of functions.

The operator $\mathcal{K}$ can be linearly extended to analytic dispersion relations given by their power series. This step is nontrivial, however, because care will have to be taken with respect to the domain of the Casimir force function: $F(L)$ is given by a power series in L^{-1} whose radius of convergence may be finite. Thus, for some dispersion relations the Casimir force may only be defined for sufficiently large separations between the plates. This is nicely consistent with the intuitive expectation that in certain media it should not make sense to consider the Casimir force for plate distances smaller than a certain finite (e.g., atomic) length scale. We will discuss possible wider ramifications at the end. To proceed, let us now write the mapping $F(L)$ in a more convenient form

$$F(L) = \frac{k_c}{L} \sum_{r=1}^{\infty} \frac{(-1)^r (2r-1)(2r-1)!}{(2\pi)^{2r}} \zeta(2r) \nu_{2r-1} \left(\frac{\pi}{k_c L} \right)^{2r-1}. \tag{16}$$

Here, $\zeta(s)$ is the Riemann zeta function. This formulation for $F(L)$ is easily obtained from (15) by using

$$B_n = \frac{2n(-1)^{n+1}}{(2\pi)^n} \cos(\pi n/2) \Gamma(n) \zeta(n). \tag{17}$$

We recall that $\zeta(r) \to 1$ very quickly as r grows. Hence, for questions related to convergence, it is sufficient to study the series with $\zeta(2r)$ set to unity. Now for a generic analytic dispersion relation, the corresponding Casimir force function series has a finite radius of convergence in $1/L$, i.e., it is defined only for plates separations larger than L_c, with

$$L_c^2 = \frac{1}{(2k_c)^2} \limsup_{r \to \infty} \left[\frac{k_c^2 (2r-1)(2r-1)!}{(\pi)} \zeta(2r) |\nu_{2r-1}| \right]^{1/r}. \tag{18}$$

An example where L_c is not zero is $f(x) = \sinh(x)$, for which $L_c = (2k_c)^{-1}$. Let us recall at this point that our calculations so far implicitly assumed that exponential regularization suffices to render the energy density finite both between the plates and outside of them. It is essential to prove that our results are independent of the exact form of the regularizing function. Let us assume now that the cut-off is realized by a general infinitely differentiable function $\gamma_\alpha(t)$ (jointly of α and t in the non-negative reals), such that $\lim_{t \searrow 0} \gamma_\alpha(t) = 1$, that $\lim_{\alpha \searrow 0} \gamma_\alpha(t) = 1$ for all t and that $\int_0^\infty \mathrm{d}x \; f(x) \gamma_\alpha [f(x)] < \infty$. The derivation of the Casimir effect presented above can then be repeated point by point using the corresponding new definition of φ_α and the same result (16) will therefore be obtained.

2. Analysis of the operator $\mathcal{K}$

Our aim is to analyse the properties of the linear operator $\mathcal{K}$ which maps dispersion relations f into Casimir force functions F, namely $\mathcal{K}: f(x) \mapsto \mathcal{F}(L)$. To this end, let us first calculate more convenient representations of $\mathcal{K}$. A first representation is obtained from (16), namely

$$\mathcal{K} = \sum_{r=1}^{\infty} \frac{k_c^2 (-1)^r (2r-1) \zeta(2r)}{\pi} \left(\frac{1}{2k_c L} \right)^{2r} \left. \frac{\mathrm{d}^{(2r-1)}}{\mathrm{d}x^{(2r-1)}} \right|_{x=0}. \tag{19}$$

For the rest of this section, we shall work with the (very good) approximation in which the zeta function is set to 1, as discussed above. It is then straightforward to check that $\mathcal{K}$ can be written as the integral operator

$$(\mathcal{K}f)(L) = \frac{k_c^2}{\pi}\,\mathrm{Im}\int_0^\infty f(\mathrm{i}x)(1-2k_cLx)\,\mathrm{e}^{-2k_cLx}\,\mathrm{d}x, \tag{20}$$

where Im stands for taking the imaginary part. Indeed, successive integrations by parts reconstruct the series of derivatives term by term, and the even powers are eliminated because they are purely real. We note that the analytic continuation of the dispersion relation to the complex axis is a mathematically safe operation since it is an entire function. It is also possible, however, to restrict attention only to the real axis, namely by neglecting the even part of the dispersion relation, as it will not contribute. Without restricting generality we can therefore assume that the dispersion relation is odd, i.e., that it can then be written in the form $f(x) = xg(x^2)$. In terms of the function g, the above integral representation then takes the form

$$(\mathcal{K}f)(L) = \frac{k_c^2}{\pi}\int_0^\infty xg(-x^2)(1-2k_cLx)\,\mathrm{e}^{-2k_cLx}\,\mathrm{d}x. \tag{21}$$

Finally, we use the properties of the Laplace transform with respect to differentiation to obtain

$$(\mathcal{K}f)(L) = \frac{k_c^2}{\pi}\left(1 + L\frac{\mathrm{d}}{\mathrm{d}L}\right)\int_0^\infty \mathrm{e}^{-2k_cLx}\,xg(-x^2)\,\mathrm{d}x. \tag{22}$$

3. The relationship between effects at different length scales

Let us now use the explicit form for $\mathcal{K}$ obtained in (22) to study how a nontrivial behaviour of the dispersion relation at one scale affects the behaviour of the corresponding Casimir force function at other scales. To this end, we note that the function g is to be evaluated at negative arguments before the Laplace transform is performed. This is important because $g(y)$ for positive y determines the dispersion relation through $f(x) = xg(x^2)$ but $g(y)$ for negative y determines the Casimir force function $\mathcal{K}f$ through (22). Clearly, g may be very close to 1 for $y > 0$ while at the same time its unique analytic extension for $y < 0$ may be far from being constant.

Let us begin by investigating the lowest order corrections to the dispersion relation f, namely $f(x) = x + \nu_2x^2 + \nu_3x^3$, with coefficients ν_2, ν_3 as large as of order one. The corresponding Casimir force function is

$$F(L) = -\frac{\pi}{24L^2} + \nu_3\frac{\pi^5}{20k_c^2L^4}. \tag{23}$$

The quadratic correction term ν_2x^2 is an even component of f and therefore does not affect the Casimir force. The next order correction significantly changes the Casimir force at very short distances where the interaction even becomes repulsive, as shown on the left of figure 1. However, $F(L)$ converges very rapidly towards the usual Casimir force function $F^{(0)}(L)$ for plate separations that are significantly larger than k_c^{-1}.

To be precise, let L_m denote the distance scale at which we may want to perform a measurement of the Casimir force. It is clear that $1/k_c$ is the length scale at which the dispersion relation starts to show significant nonlinearity. The dimensionless parameter $\sigma = (k_cL_m)^{-1}$ thus quantifies how far the scales $1/k_c$ and L_m are apart. In terms of σ, the relative size of the effect of the x^3 modification to the dispersion relation on the Casimir force is then found to be

$$\frac{F^{(0)}(L) - F(L)}{F^{(0)}(L)} = \nu_3\frac{6\pi^4}{5}\sigma^2 \ll 1. \tag{24}$$

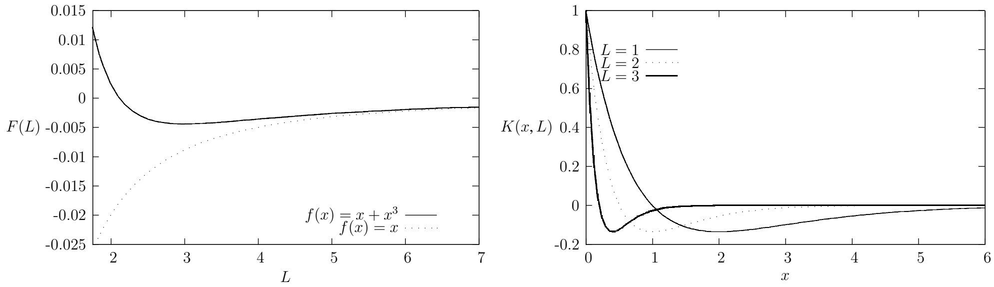

Figure 1. The Casmir force for the lowest order corrections (left) and the integral kernel K (right). L is in units of k_c^{-1}.

We observe that the effect on the Casimir force is suppressed quadratically with the ratio, σ, of the dispersion modification scale $1/k_c$ and the Casimir force measurement scale L_m. This is consistent with the expectation that a modification of the dispersion at very short length scales only very weakly affects the Casimir force at longer length scales.

Naively, one might expect that higher order corrections to the dispersion at short lengths contribute even less to the Casimir force at larger length scales. Indeed, dispersion relation terms $\propto x^{2r-1}$ are mapped to Casimir force terms $\propto \sigma^{2r-1}$, so that it would seem that higher order corrections get exponentially suppressed. Interestingly, however, this is not the case. The exact mapping (16) contains an extra factorial $(2r-1)!$ that grows faster than exponentially. The relative correction for $f(x) = x + \nu_{2r-1}x^{2r-1}$ is

$$\frac{F^{(0)}(L) - F(L)}{F^{(0)}(L)} = \nu_{2r-1}\frac{(-1)^{r-1}(2r-1)\zeta(2r)}{4\pi^2}(2r-1)!\left(\frac{\sigma}{2}\right)^{2r-2}. \qquad (25)$$

It is straightforward to apply Stirling's formula for the factorial, $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ for $n \gg 1$, in order to calculate how large r needs to be for the factorial amplification to overcome the exponential suppression. We find that a correction term $\nu_{2r-1}x^{2r-1}$ with $\nu_{2r-1} \approx 1$ in the dispersion relation leads to a relative change of order one in the Casimir force at the infrared scale L_m if r is of the order σ^{-1}. This means that, for example, a modified dispersion relation of the form $f(x) = x + \nu_{2r-1}x^{2r-1}$, say with $r \approx 10^5$ and $\nu_{2r-1} \approx 1$, is virtually indistinguishable from the linear dispersion relation $f(x) = x$ at all scales up to the length scale $1/k_c$, and yet it does lead to a modification of the Casimir force which is of order one even at a measurement length scale L_m with five orders of magnitude longer than $1/k_c$. We conclude that even though the first-order terms contribute extremely little to the Casimir force, very high order corrections to the dispersion relations can contribute significantly to the Casimir force, in fact, the more so the larger r is.

Actual dispersion relations might be given by a series $f(x) = x + \sum_{n=2}^{\infty}\nu_n x^n$ and therefore contain terms $\nu_{2r-1}x^{2r-1}$ for arbitrarily large r. At the same time, the prefactors ν_n must of course obey $\nu_n \to 0$ as $n \to \infty$ because this is a necessary condition for the convergence of the series. We conclude that it is this competition between the decay of the coefficients ν_{2r-1} and the increasing influence on the Casimir effect of terms x^{2r-1}, for $r \to \infty$, which decides whether or not a dispersion relation that is significantly nonlinear only below a length scale $1/k_c$ does or does not lead to an appreciable effect on the Casimir force at significantly larger plate separations, $L_m \gg 1/k_c$. Let us now explicitly investigate this competition. To this end, we can conveniently make use of the representation of $\mathcal{K}$ in terms of the Laplace transform (21).

Firstly, we recall that for $f(x) = xg(x^2)$ the mapping involves evaluating g on the negative real axis. g being an entire function, its behaviour on the positive half-axis fully determines it on the negative half-axis too, which implies as it should be that the dispersion relations do determine the corresponding Casimir forces uniquely. Secondly, however, there exist entire functions g which are arbitrarily close to 1 for $0 < y < 1$ and which nevertheless reach arbitrarily large values on the negative half-axis. Such functions do not noticeably affect the dispersion relation down to the length scale $1/k_c$ but they do strongly affect the Casimir force at length scales L_m that can be arbitrarily larger than $1/k_c$. These are the dispersion relations with

$$g(y) = 1 + h(y), \tag{26}$$

where the function h obeys $h(y) \approx 0$ for $y \in (0, 1)$ while exhibiting large $|h(y)|$ in some range of negative values of y. The correction to the force $\Delta F = F - F^{(0)}$ in terms of the correction h to the dispersion relation can again be read from (21):

$$\Delta F(L) = \frac{k_c^2}{\pi} \int_0^\infty x h(-x^2)(1 - 2k_c L x)\, \mathrm{e}^{-2k_c L x}\, \mathrm{d}x. \tag{27}$$

The integral kernel

$$K(x, L) = (1 - 2k_c L x)\, \mathrm{e}^{-2k_c L x} \tag{28}$$

is positive for $x < (2k_c L)^{-1}$, negative for $x > (2k_c L)^{-1}$ and exponentially decreases to zero for $x \gg (2k_c L)^{-1}$. (Note that the integral of the kernel over all $x \in [0, \infty)$ is 0, which expresses the fact that the Casimir force does not depend on the absolute value of the energy.) Thus, for a fixed plate separation L, what matters most for the Casimir force is the behaviour of $h(y)$ from $y = 0$ to about $y \approx -(k_c L)^{-2}$. As we increase L, the interval $y \in (-(k_c L)^{-2}, 0)$ on which the integral kernel K is mostly supported is shrinking, as can be seen on the right of figure 1.

Thus, there is a significant effect on the Casimir force at large plate separations, $L = L_m \gg 1/k_c$, if the function h is either of order 1 in this small interval close to the origin or it must be exponentially large (so as to compensate the exponential suppression in K) in some interval to the left of $-(k_c L)^{-2}$. Of course, both are possible. There are entire functions h which possess either one of these behaviours on the negative half-axis. These are arbitrarily close to 0 for $0 < y < 1$, so as to leave the dispersion relation virtually unchanged at large length scales and yet they do significantly affect the Casimir force at large L_m.

There is even the extreme case of functions, h, whose corresponding dispersion relation f is arbitrarily little affected at all scales while the Casimir force function is arbitrarily much affected at any desired long length scale. An example is the case where h is a Gaussian which is centred around a low negative value $y_0 < 0$ while being so sharply peaked that its tail into the positive half-axis is negligibly small. The function that enters into the calculation of the Casimir force, $xg(-x^2)$, then features the low-x spike of the Gaussian, implying by our above consideration that the Casimir force is affected at large plates separation. At the same time, the dispersion relation itself, $xg(x^2)$, is virtually unaffected for all x.

4. Conclusions and outlook

We calculated the operator $\mathcal{K}$ which maps general dispersion relations into the corresponding Casimir force functions and found that $\mathcal{K}$ is essentially a Laplace transform. The analysis of $\mathcal{K}$ showed that, as expected, modifications to the vacuum dispersion relation below some length scale $1/k_c$ generally only negligibly affect the Casimir force function at plate distances L_m that are significantly larger than $1/k_c$. We also found, however, that for fine-tuned dispersion relations their nonlinearities at a scale $1/k_c$ can strongly affect the Casimir force at arbitrarily larger length scales L_m. It should be interesting, of course, to generalize the present analysis to electrodynamics in three dimensions and apply it to dispersion relations of real media.

Finally, let us speculate that it also should be interesting to calculate, in a similar study of a quantum vacuum effect, the operator $\mathcal{K}$ which in inflation maps generalized dispersion relations of the inflation into the corresponding inflationary perturbation spectra. This is of interest because generalized dispersion relations could encode Planck scale physics, while the inflationary perturbation spectrum is understood to be measurable in the cosmic microwave background (CMB). Calculations have been performed so far only for particular dispersion relations, and there is a debate as to the magnitude of the generically to be expected effect at measurable scales in the CMB [5, 6]. The relevant dimensionless σ is the ratio between the Planck length and the Hubble length during inflation. In inflation, σ is around $\sigma \approx 10^{-5}$, indicating that any effects would likely be suppressed by a factor of at least σ. If the operator $\mathcal{K}$ of inflation behaves similar to that in the Casimir effect, the effect in inflation could be larger for some fine-tuned dispersion relations, possibly bringing them closer to observability.

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